Finite Element Methods

Week No-03

Introduction to Matrix Structural Analysis

- Classical Methods vs. Matrix Methods.
- Planar Frame Member, Nodal Displacements & Nodal Forces in Local Coordinates
- Truss Analysis by Stiffness Matrix Method
  - Truss Member Stiffness Relations in Local Coordinates
  - Member Stiffness Relations in Global Coordinates
  - Structure Stiffness Relations
  - Procedure for Analysis
- Frame & Beam Analysis by Stiffness Matrix Method
  - Analytical Model for Planar Frame Structure
  - Global & Local Coordinate Systems
  - Member Stiffness Relations in Local Coordinates
  - Stiffness Matrix of 2D Frame Member in Global Coordinates
  - Member Stiffness Relations in Global Coordinates
  - Structure Stiffness Relations

Classical Methods vs Matrix Methods

Classical Methods	Matrix Methods
Help to understand the structural behavior & the principles of structural Analysis	Simplify the overall picture of Structural Analysis
Time consuming for the analysis of large systems	Time saving as being computerized
Vary according to the structure type	Less varying

Analytical Model for Planar Frame Structures

In the matrix stiffness method of analysis the structure is considered to be an assemblage of straight & prismatic members connected at their ends to joints.

A member is defined as a part of the structure for which the member force-displacement relations to be used in the analysis are valid.

A member is a straight part of the structure with constant cross-sectional properties (*I* & *A*) along its length

A joint is defined as a structural part of infinitesimal size to which the member ends are connected

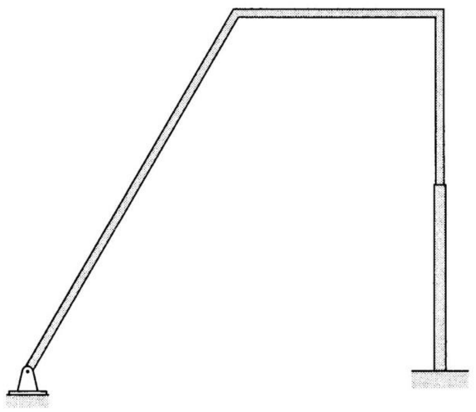
Members & Joints are also referred as Elements & Nodes

The model is a line diagram of the structure where joints & members are identified by numbers.

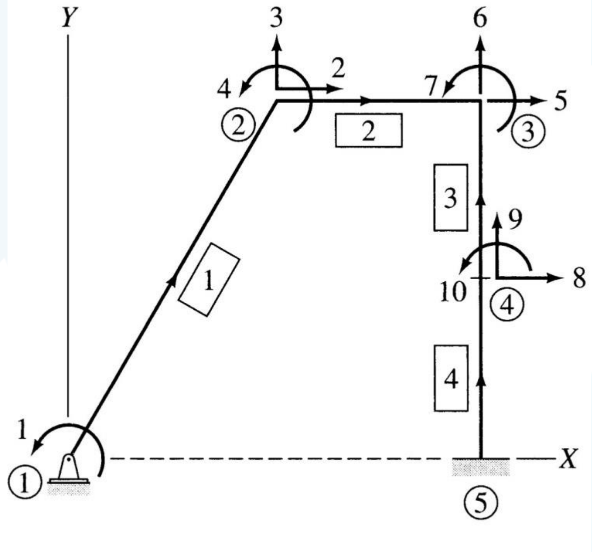
Joint numbers are enclosed within circles

Member numbers are enclosed within rectangles

Arrow from the member begging joint to its end joint



(a) Actual Frame



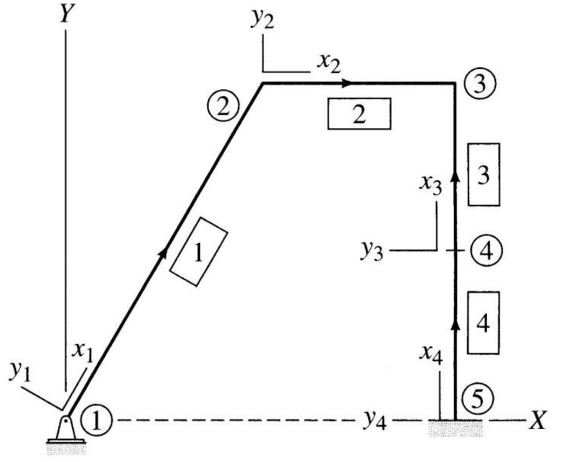
(b) Analytical Model

Analytical Model for Planar Frame Structures

Global & Local Coordinate Systems

The global (or structure) coordinate system *XYZ* is a right-handed Cartesian or rectangular one starting at any arbitrary point.

The local (or member) coordinate system *xyz,* is a right-handed Cartesian coordinate system starting at the begging joint of the member

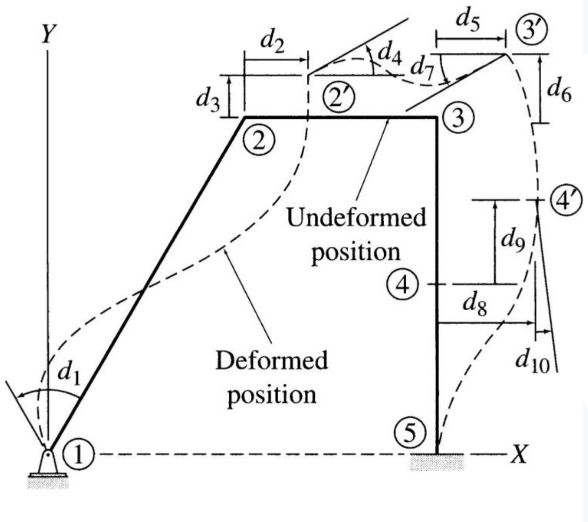


(c) Global and Local Coordinate Systems

Degrees of Freedom

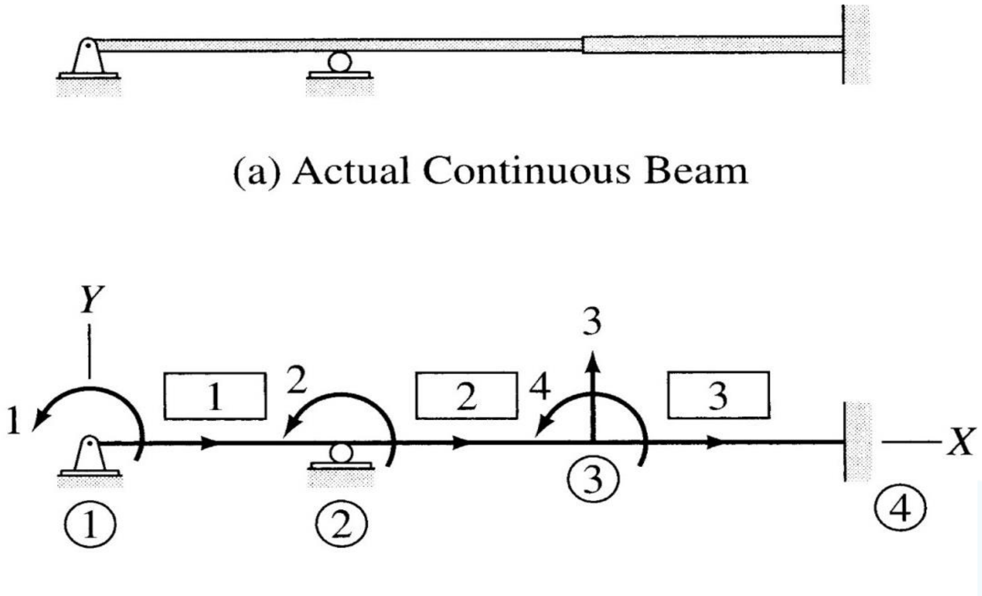
The **degrees of freedom** of a structure are the independent joint displacements (translations & rotations) that are necessary to specify the deformed shape of the structure when subjected to an arbitrary loading.

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ \vdots \\ d_9 \\ d_{10} \end{bmatrix}$$



(d) Degrees of Freedom

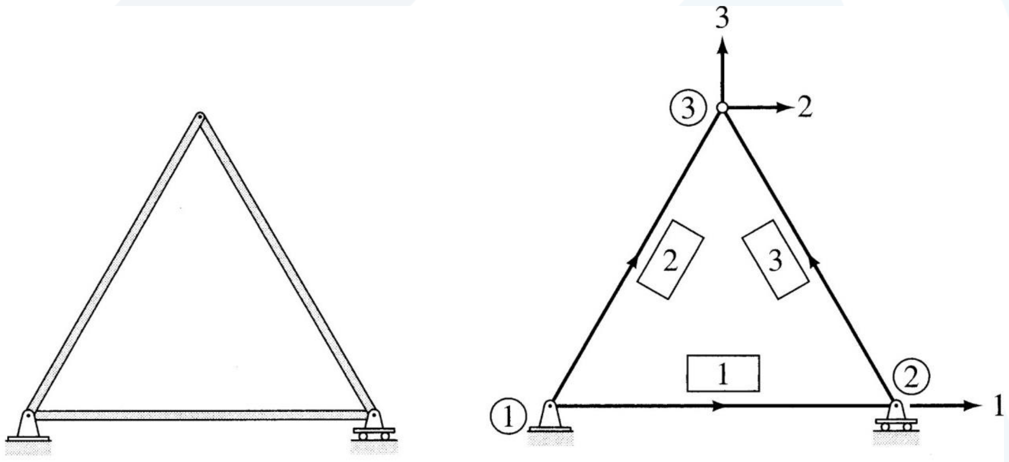
$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \\ d_4 \end{bmatrix}$$



(a) Actual Continuous Beam

(b) Analytical Model and Degrees of Freedom

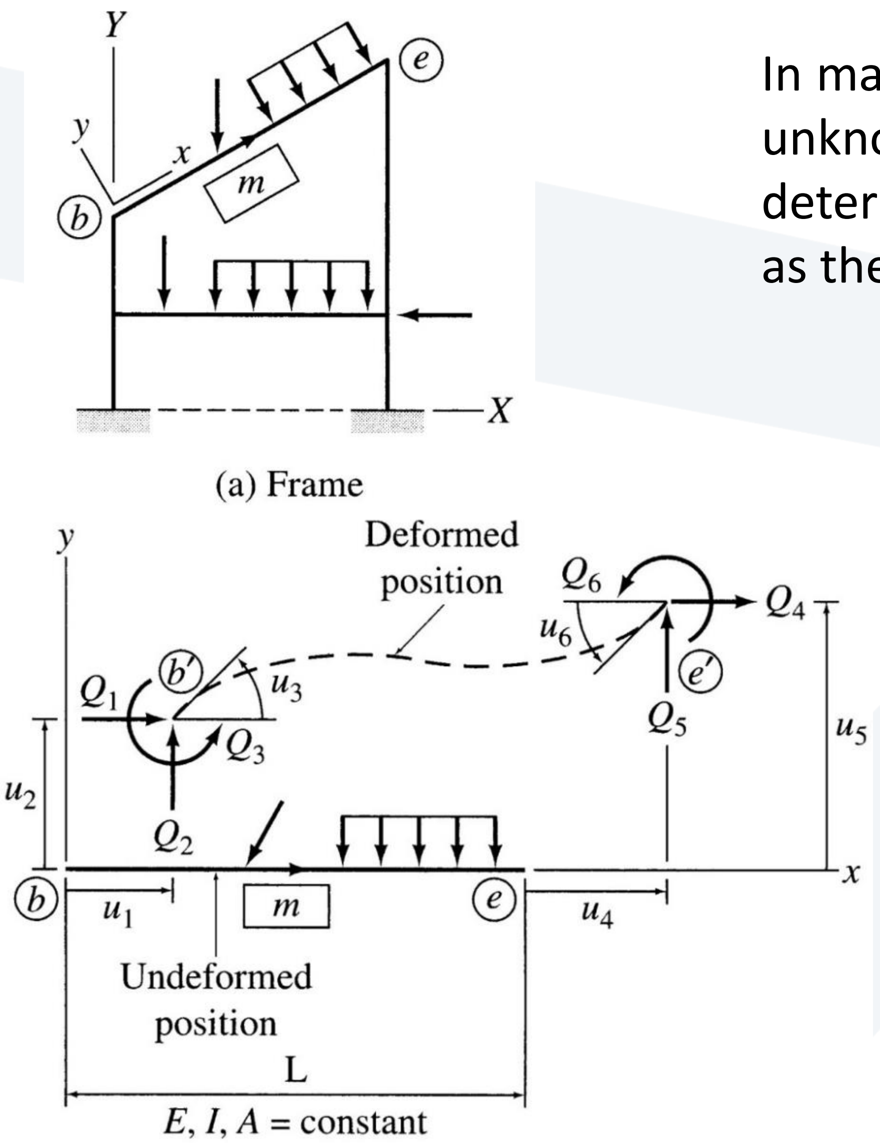
$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \\ d_3 \end{bmatrix}$$



(a) Actual Truss

(b) Analytical Model and Degrees of Freedom

Member Stiffness Relations in Local Coordinates



(a) Frame

(b) Frame Member — Local Coordinates

In matrix stiffness method of structural analysis, the primary unknowns, the joint displacements of the structure are determined by solving a system of simultaneous equations as the following one

$$\mathbf{Sd} = \bar{\mathbf{P}}$$

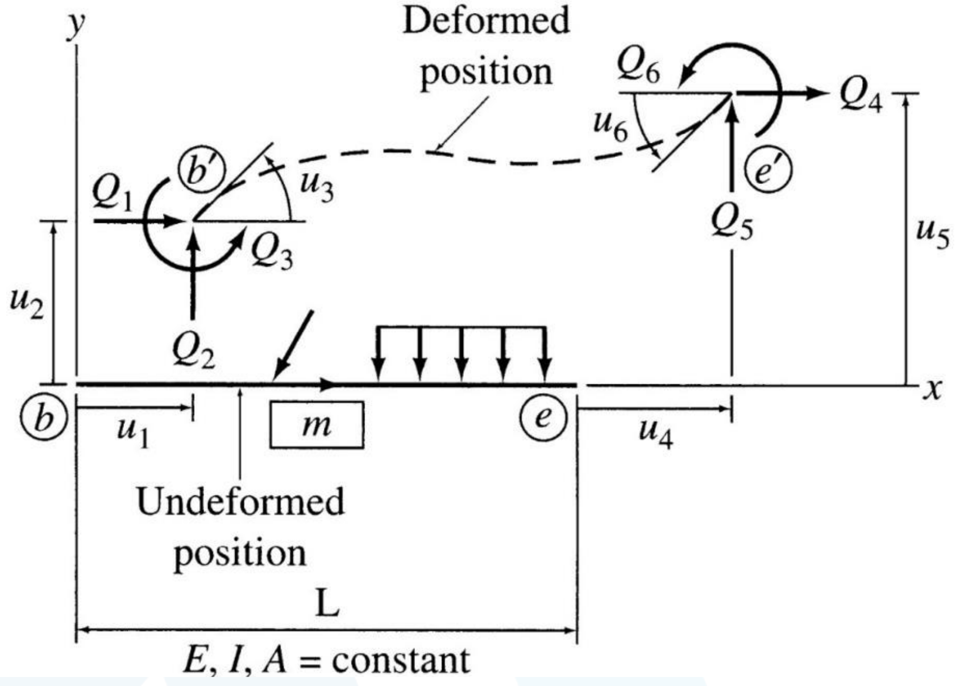
d, the joint displacement vector

$\bar{\mathbf{P}}$**,** the joint external load vector

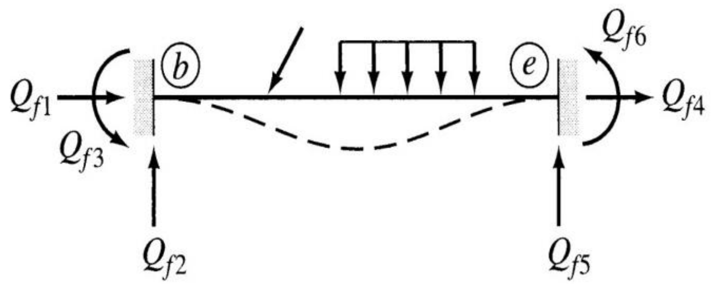
S, the structure stiffness matrix, obtained by assembling the stiffness matrices $\mathbf{k}^m$, for the individual members which express the end forces Q_i of the member in term of its joint displacements u_i

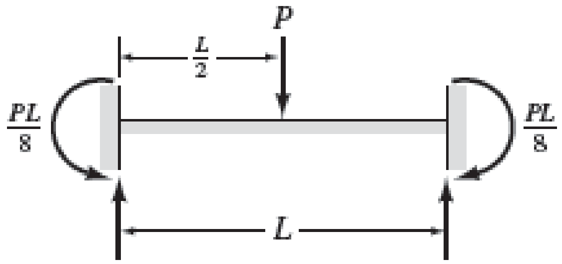
$$\mathbf{Q} = \mathbf{ku} + \mathbf{Q}_f$$

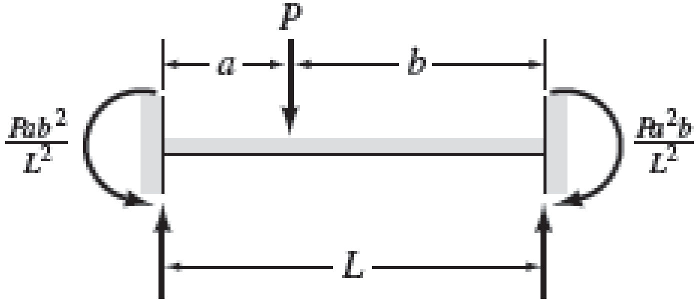
Planar Frame Members Local Coordinates

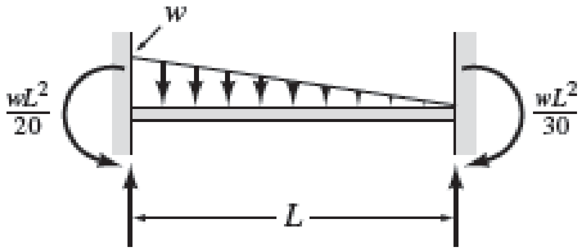


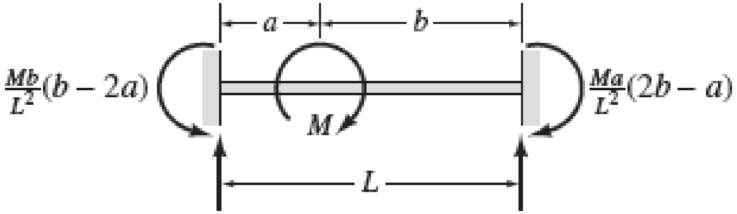
$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{bmatrix} = \begin{bmatrix} k_{11} & k_{12} & k_{13} & k_{14} & k_{15} & k_{16} \\ k_{21} & k_{22} & k_{23} & k_{24} & k_{25} & k_{26} \\ k_{31} & k_{32} & k_{33} & k_{34} & k_{35} & k_{36} \\ k_{41} & k_{42} & k_{43} & k_{44} & k_{45} & k_{46} \\ k_{51} & k_{52} & k_{53} & k_{54} & k_{55} & k_{56} \\ k_{61} & k_{62} & k_{63} & k_{64} & k_{65} & k_{66} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \\ Q_{f5} \\ Q_{f6} \end{bmatrix}$$

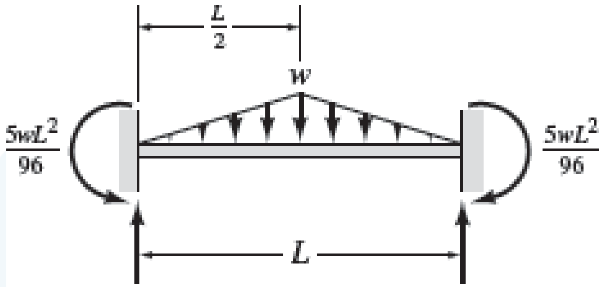

Q_{f1}
Q_{f3}
Q_{f2}
b
e
Q_{f6}
Q_{f4}
Q_{f5}

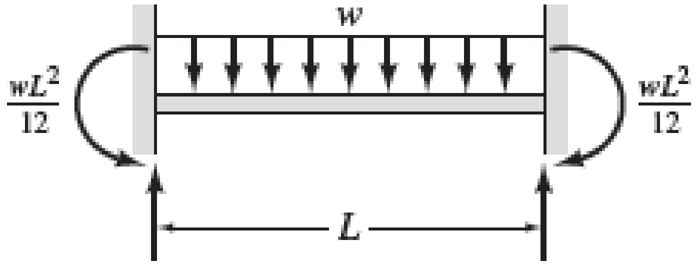

P
$\frac{L}{2}$
$\frac{PL}{8}$
$\frac{PL}{8}$
L

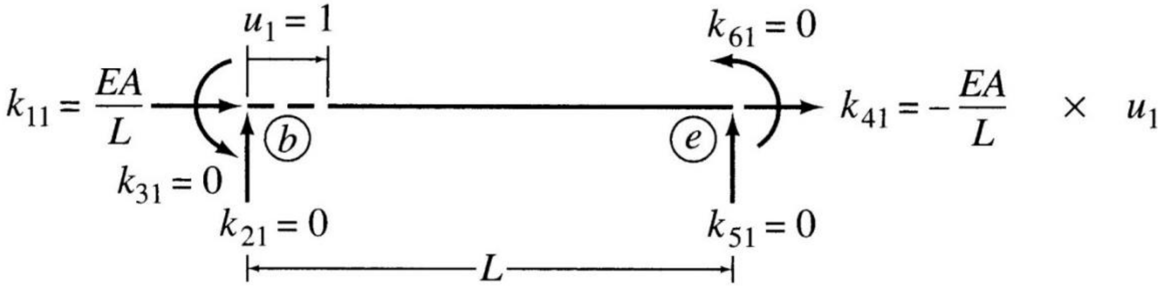

P
a
b
$\frac{Pab^2}{L^2}$
$\frac{Pa^2b}{L^2}$
L


w
$\frac{wL^2}{20}$
$\frac{wL^2}{30}$
L


a
b
$\frac{Mb}{L^2}(b-2a)$
$\frac{Ma}{L^2}(2b-a)$
M
L


$\frac{L}{2}$
w
$\frac{5wL^2}{96}$
$\frac{5wL^2}{96}$
L


w
$\frac{wL^2}{12}$
$\frac{wL^2}{12}$
L

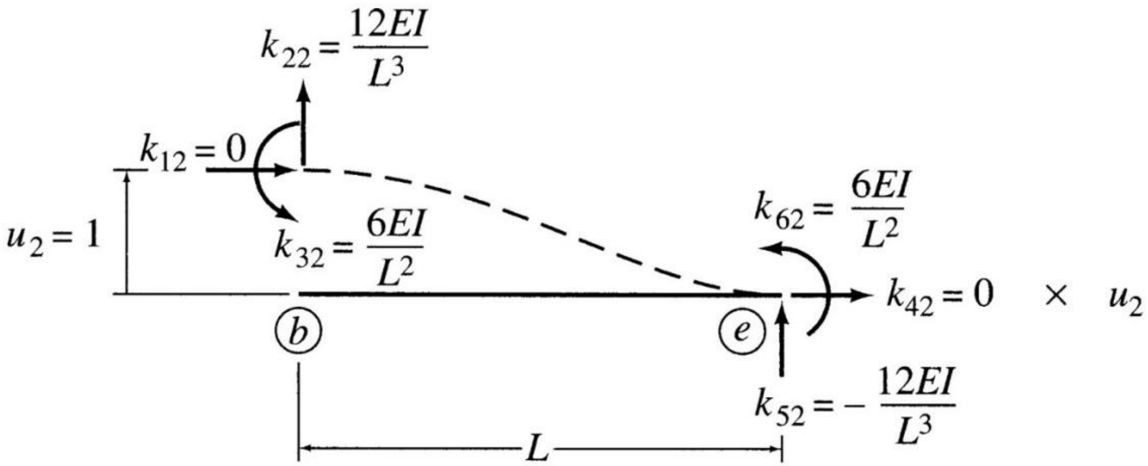


$\Delta = L\varepsilon = u_1$, Compatibility Equation $\quad$ $Q_1 = \sigma A$, Internal Equilibrium Equation

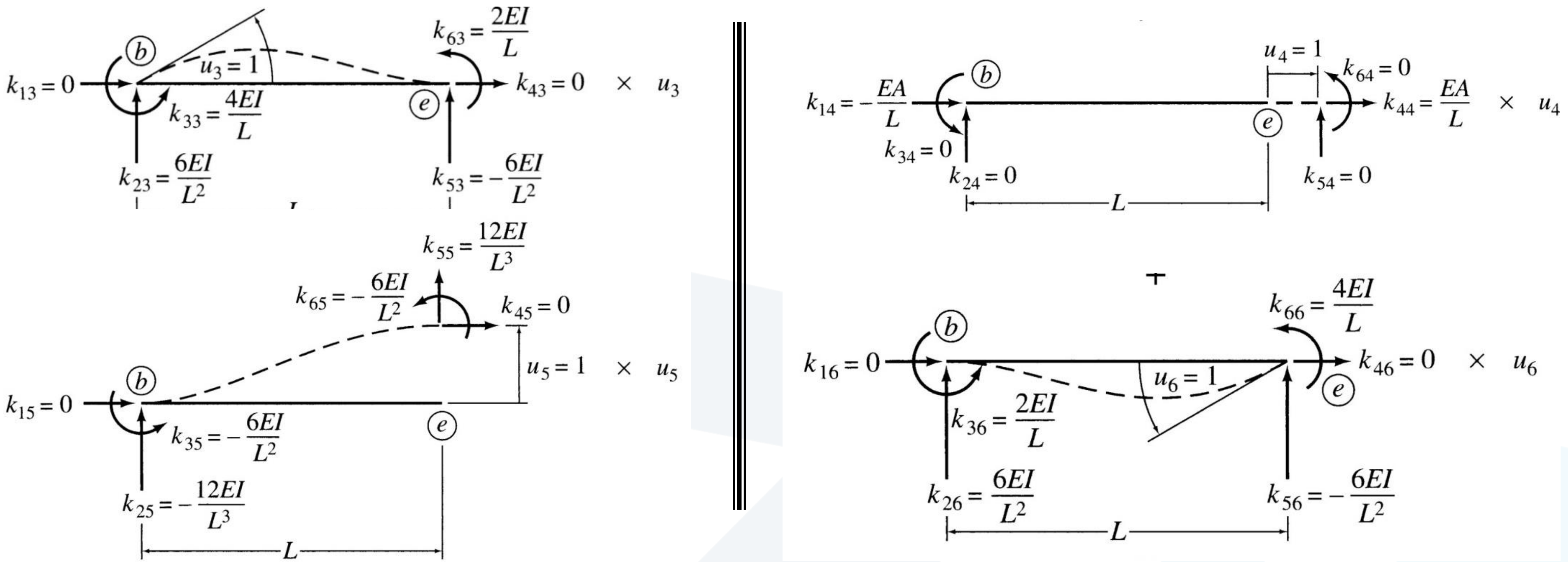
$\sigma = E\varepsilon$, Constitutive Equation

$$Q_1 = (EA/L)\, u_1 = k_{11} u_1$$

$Q_1 = -Q_4$, External Equilibrium Equation $\quad k_{11} = -k_{41} = EA/L \quad k_{21} = k_{31} = k_{51} = k_{61} = 0$



$k_{22} = -k_{52} = 12EI/L^3 \quad k_{12} = k_{42} = 0 \quad k_{32} = k_{62} = 6EI/L^2$



$$
\mathbf{k}=\frac{EI}{L^3}\begin{bmatrix}
\frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\
0 & 12 & 6L & 0 & -12 & 6L \\
0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\
-\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\
0 & -12 & -6L & 0 & 12 & -6L \\
0 & 6L & 2L^2 & 0 & -6L & 4L^2
\end{bmatrix}
$$

2D Frame members

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} \frac{AL^2}{I} & 0 & 0 & -\frac{AL^2}{I} & 0 & 0 \\ 0 & 12 & 6L & 0 & -12 & 6L \\ 0 & 6L & 4L^2 & 0 & -6L & 2L^2 \\ -\frac{AL^2}{I} & 0 & 0 & \frac{AL^2}{I} & 0 & 0 \\ 0 & -12 & -6L & 0 & 12 & -6L \\ 0 & 6L & 2L^2 & 0 & -6L & 4L^2 \end{bmatrix}$$

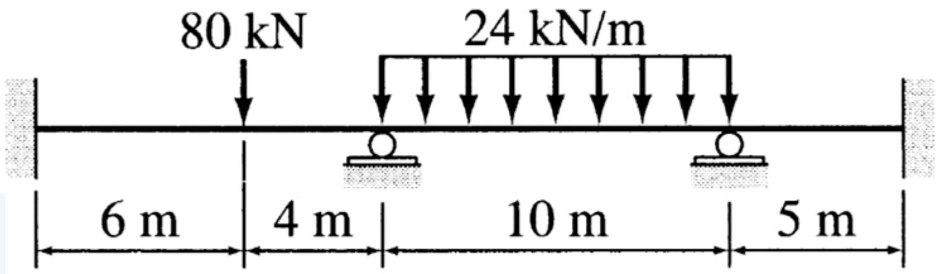
Truss Members

$$\mathbf{k} = \frac{EA}{L}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$$

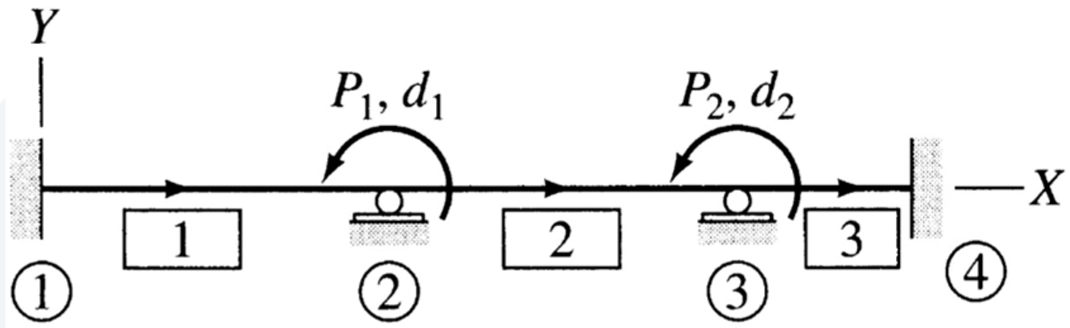
Continuous Beam Members

$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6 & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6 & 2l^2 & -6L & 4L^2 \end{bmatrix}$$

Determine the reactions and the member end forces for the three-span continuous beam using the matrix stiffness method



(a) Continuous Beam



(b) Analytical Model

Solution:

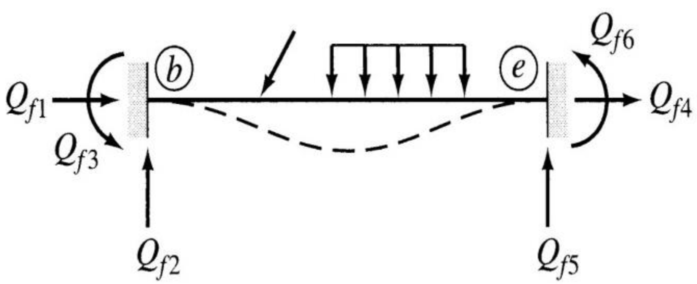
***Degrees of Freedom*:**

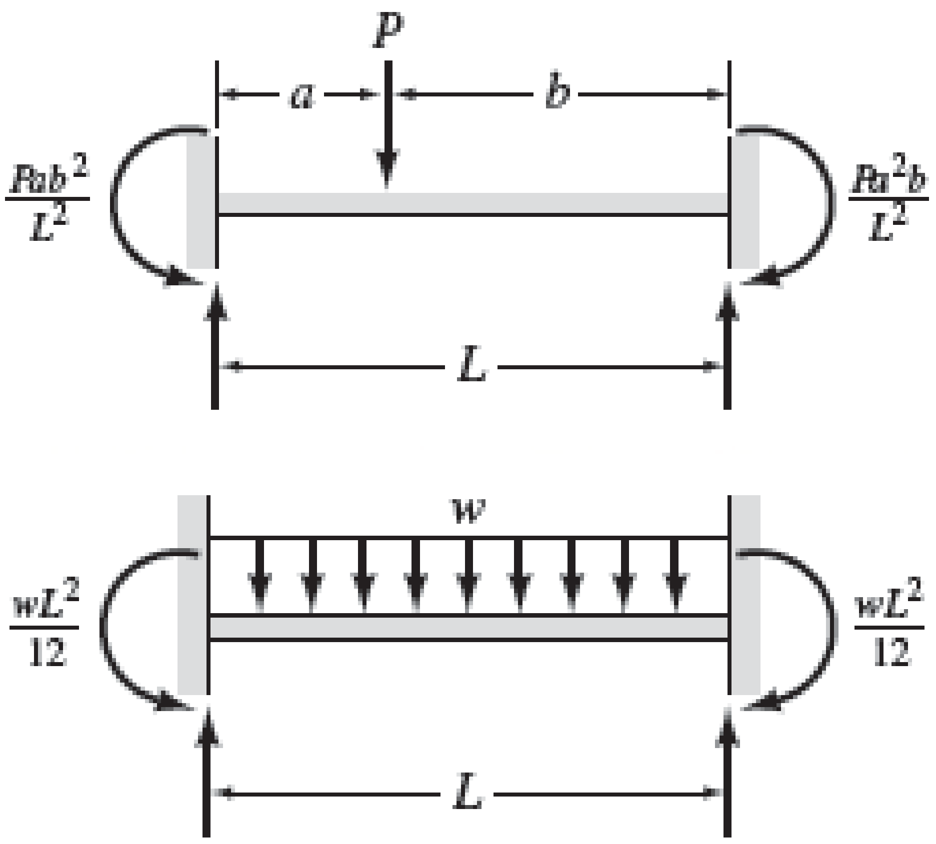
From the analytical model, d_1 & d_2, the two rotations at nodes 2 & 3, are the two unknown degrees of freedom

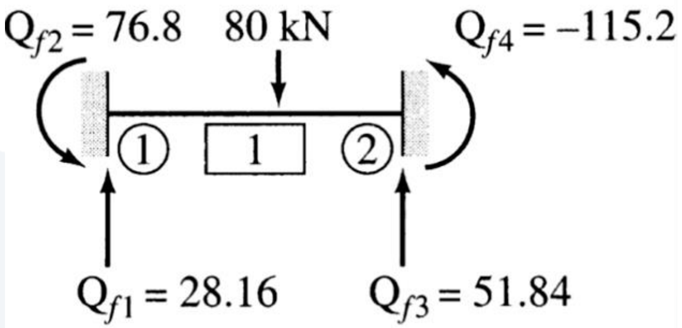
The element stiffness matrix

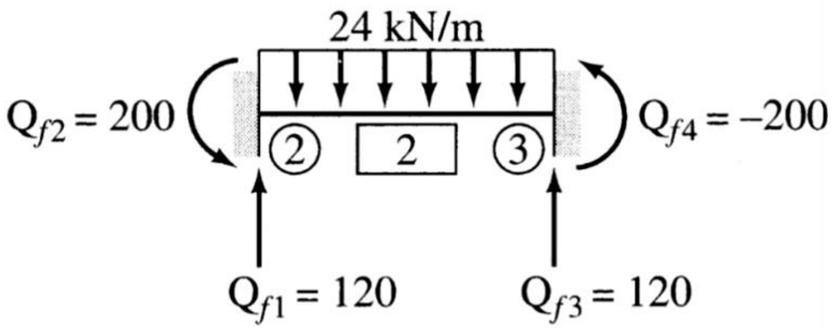
$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6 & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix}$$

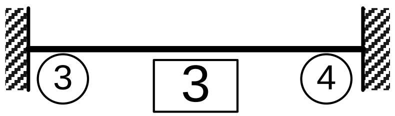
The Local & Global Coo. Sys. are identical


Q_{f6}
Q_{f1}
Q_{f4}
Q_{f3}
b
e
Q_{f2}
Q_{f5}


P
a
b
$\frac{Pab^2}{L^2}$
$\frac{Pa^2b}{L^2}$
L
w
$\frac{wL^2}{12}$
$\frac{wL^2}{12}$
L



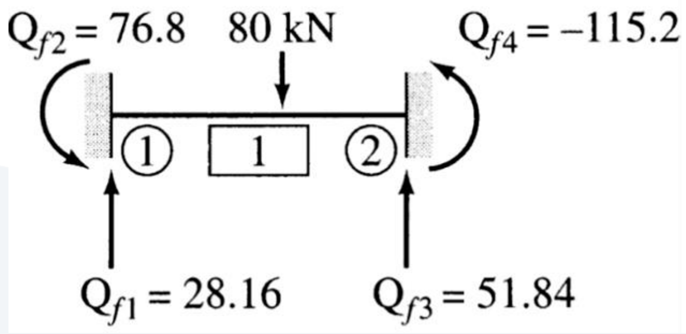


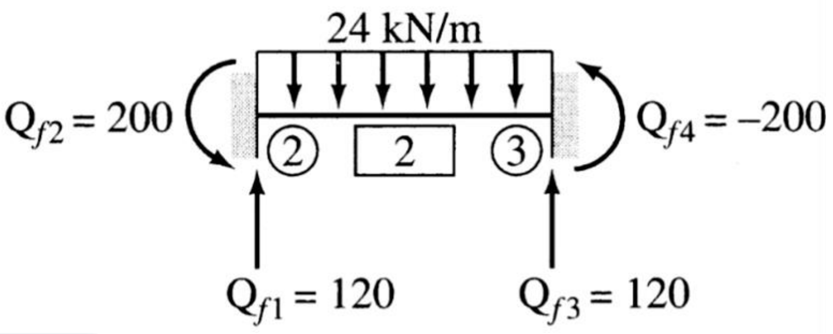


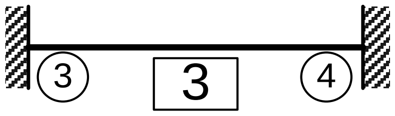
$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix}$$

Member 1: using L=10m & the fixed-end forces from the table

$$\mathbf{K}_1 = \mathbf{k}_1 = EI \begin{array}{c} \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} \\ \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \qquad \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} 28.16 \\ 76.80 \\ 51.84 \\ -115.2 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}$$



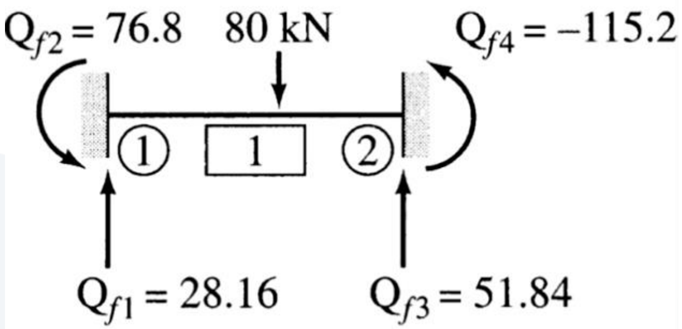


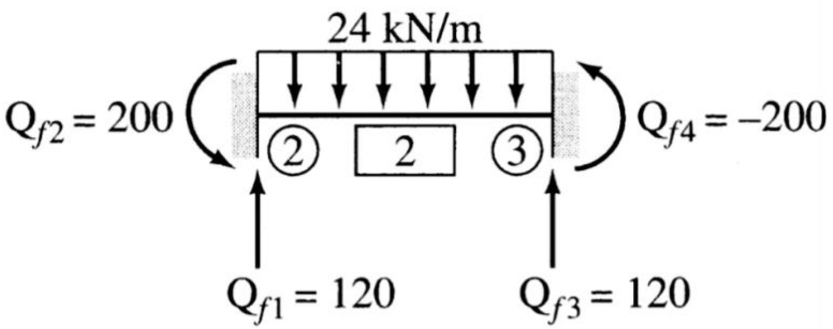


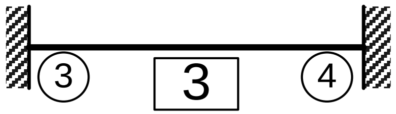
$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix}$$

Member 2: using L=10m & the fixed-end forces from the table

$$\mathbf{K}_2 = \mathbf{k}_2 = EI \begin{array}{c} \begin{array}{cccc} 0 & 1 & 0 & 2 \end{array} \\ \left[\begin{array}{cccc} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{array}\right] \end{array} \begin{array}{c} 0 \\ 1 \\ 0 \\ 2 \end{array} \qquad \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} 120 \\ 200 \\ 120 \\ -200 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 0 \\ 2 \end{array}$$







$$\mathbf{k} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix}$$

Member 3: using L=5m & the fixed-end forces from the table

$$\mathbf{K}_3 = \mathbf{k}_3 = EI \begin{array}{c} \begin{array}{cccc} 0 & 2 & 0 & 0 \end{array} \\ \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.8 & -0.048 & 0.4 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.048 & 0.4 & -0.24 & 0.8 \end{bmatrix} \end{array} \begin{array}{c} 0 \\ 2 \\ 0 \\ 0 \end{array} \qquad \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \begin{array}{c} 0 \\ 2 \\ 0 \\ 0 \end{array}$$

$$\mathbf{K}_1 = \mathbf{k}_1 = EI \begin{array}{cc} \begin{array}{cccc} 0 & 0 & 0 & 1 \end{array} & \\ \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} & \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array} \end{array} \qquad \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} 28.16 \\ 76.80 \\ 51.84 \\ -115.2 \end{bmatrix} \begin{array}{c} 0 \\ 0 \\ 0 \\ 1 \end{array}$$

$$\mathbf{K}_2 = \mathbf{k}_2 = EI \begin{array}{cc} \begin{array}{cccc} 0 & 1 & 0 & 2 \end{array} & \\ \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.06 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.06 & 0.2 & -0.06 & 0.4 \end{bmatrix} & \begin{array}{c} 0 \\ 1 \\ 0 \\ 2 \end{array} \end{array} \qquad \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix} = \begin{bmatrix} 120 \\ 200 \\ 120 \\ -200 \end{bmatrix} \begin{array}{c} 0 \\ 1 \\ 0 \\ 2 \end{array}$$

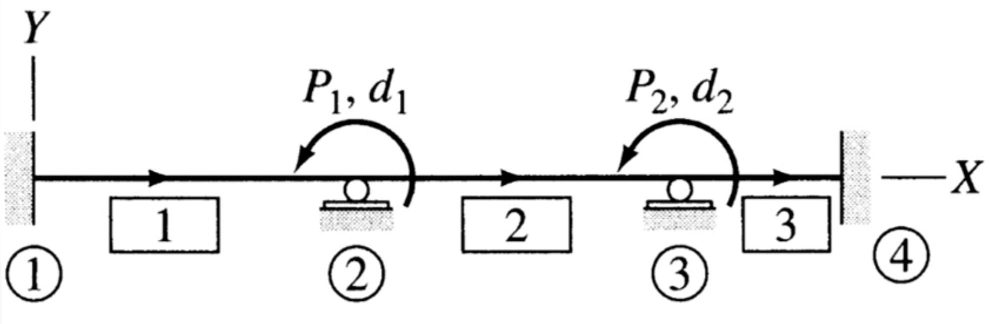
$$\mathbf{K}_3 = \mathbf{k}_3 = EI \begin{array}{cc} \begin{array}{cccc} 0 & 2 & 0 & 0 \end{array} & \\ \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.8 & -0.048 & 0.4 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.048 & 0.4 & -0.24 & 0.8 \end{bmatrix} & \begin{array}{c} 0 \\ 2 \\ 0 \\ 0 \end{array} \end{array}$$

$$EI \begin{bmatrix} 0.8 & 0.2 \\ 0.2 & 1.2 \end{bmatrix} \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \begin{bmatrix} -84.8 \\ 200 \end{bmatrix}$$

$$\mathbf{d} = \begin{bmatrix} d_1 \\ d_2 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} -154.09 \\ 192.35 \end{bmatrix}$$

Member End Displacements and End Forces

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix}$$



(b) Analytical Model

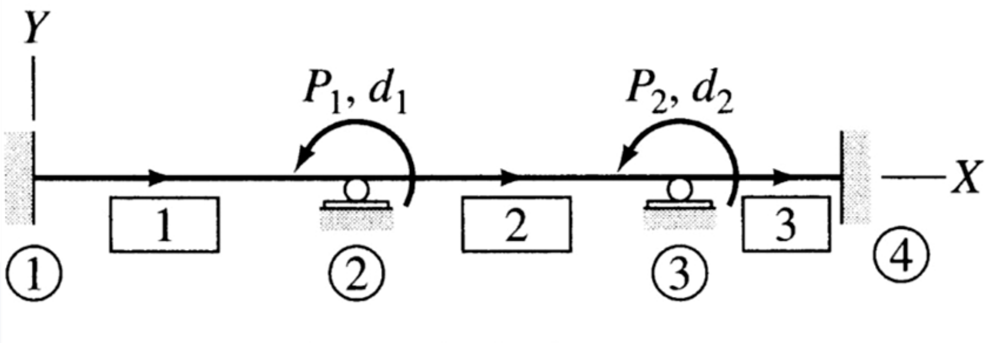
Member 1:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} 0 \\ 0 \\ 0 \\ 1 \end{matrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ d_1 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -154.09 \end{bmatrix}$$

$$\mathbf{F}_1 = \mathbf{Q}_1 = EI \begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.006 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.006 & 0.2 & -0.06 & 0.4 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 0 \\ 0 \\ -154.09 \end{bmatrix} + \begin{bmatrix} 28.16 \\ 76.80 \\ 51.84 \\ -115.2 \end{bmatrix} = \begin{bmatrix} 18.91\text{kN} \\ 45.98\text{kN}-\text{m} \\ 61.09\text{kN} \\ -176.84\text{kN}-\text{m} \end{bmatrix}$$

Member End Displacements and End Forces

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix}\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix}$$



(b) Analytical Model

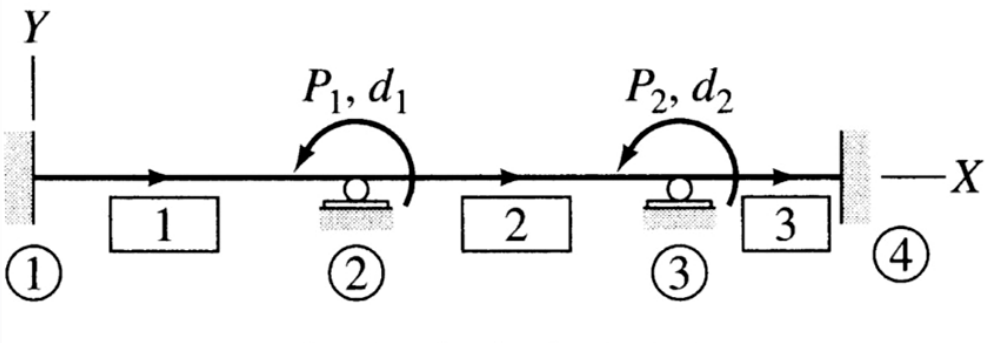
Member 2:

$$\mathbf{u}_2 = \mathbf{v}_2 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix}\begin{matrix} 0 \\ 1 \\ 0 \\ 2 \end{matrix} = \begin{bmatrix} 0 \\ d_1 \\ 0 \\ d_2 \end{bmatrix} = \frac{1}{EI}\begin{bmatrix} 0 \\ -154.09 \\ 0 \\ 192.35 \end{bmatrix}$$

$$\mathbf{F}_2 = \mathbf{Q}_2 = EI\begin{bmatrix} 0.012 & 0.06 & -0.012 & 0.06 \\ 0.06 & 0.4 & -0.006 & 0.2 \\ -0.012 & -0.06 & 0.012 & -0.06 \\ 0.006 & 0.2 & -0.06 & 0.4 \end{bmatrix}\frac{1}{EI}\begin{bmatrix} 0 \\ -154.09 \\ 0 \\ 192.35 \end{bmatrix} + \begin{bmatrix} 120 \\ 200 \\ 120 \\ -200 \end{bmatrix} = \begin{bmatrix} 122.3\text{kN} \\ 176.83\text{kN}-\text{m} \\ 117.7\text{kN} \\ -153.88\text{kN}-\text{m} \end{bmatrix}$$

Member End Displacements and End Forces

$$\begin{bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{bmatrix} = \frac{EI}{L^3} \begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2l^2 & -6L & 4L^2 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix} + \begin{bmatrix} Q_{f1} \\ Q_{f2} \\ Q_{f3} \\ Q_{f4} \end{bmatrix}$$

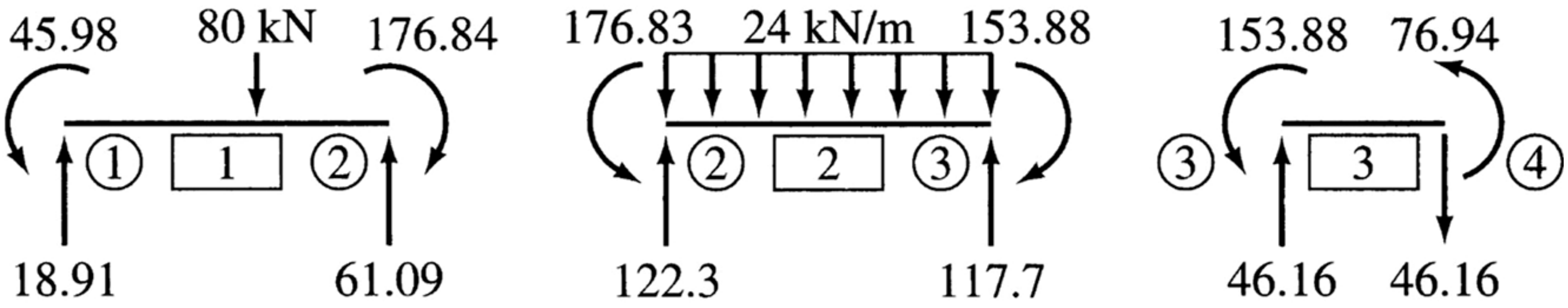


(b) Analytical Model

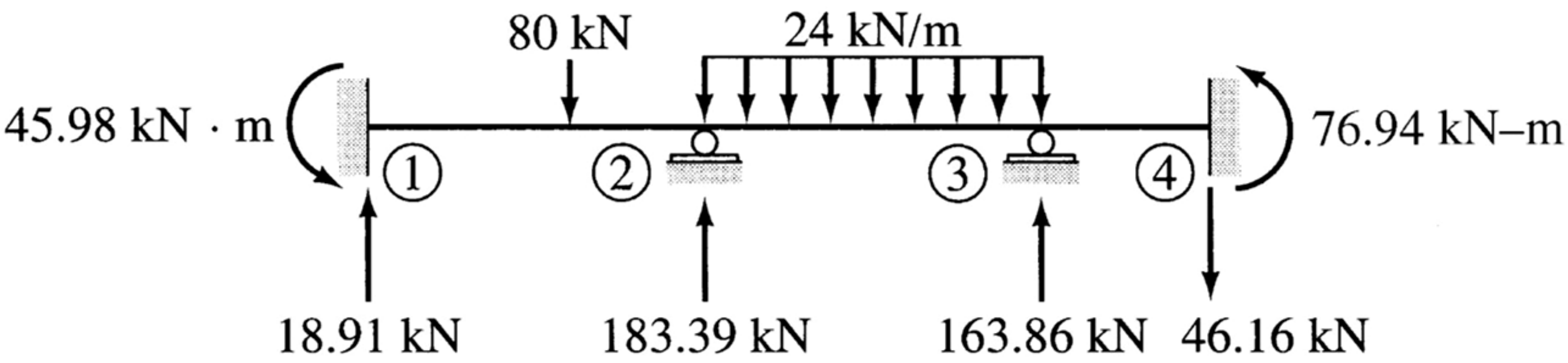
Member 3:

$$\mathbf{u}_3 = \mathbf{v}_3 = \begin{bmatrix} v_1 \\ v_2 \\ v_3 \\ v_4 \end{bmatrix} \begin{matrix} 0 \\ 2 \\ 0 \\ 0 \end{matrix} = \begin{bmatrix} 0 \\ d_2 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{EI} \begin{bmatrix} 0 \\ 192.35 \\ 0 \\ 0 \end{bmatrix}$$

$$\mathbf{F}_3 = \mathbf{Q}_3 = EI \begin{bmatrix} 0.096 & 0.24 & -0.096 & 0.24 \\ 0.24 & 0.8 & -0.048 & 0.4 \\ -0.096 & -0.24 & 0.096 & -0.24 \\ 0.048 & 0.4 & -0.24 & 0.8 \end{bmatrix} \frac{1}{EI} \begin{bmatrix} 0 \\ 192.35 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 46.16\text{kN} \\ 153.88\text{kN}-\text{m} \\ -46.16\text{kN} \\ 76.94\text{kN}-\text{m} \end{bmatrix}$$



(e) Member End Forces



(f) Support Reactions